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USE OF GENERALIZED MASS IN THE INTERPRETATION OF DYNAMIC RESPONSE OF BENDING–TORSION COUPLED BEAMS

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The interpretation of mode shapes and dynamic response of bending-torsion coupled beams is assessed by using the concept of generalized mass. In the first part of this investigation, the free vibratory motion of bending–torsion coupled beams is studied in detail. The conventional method of interpreting the normal modes of vibration consisting of bending displacements and torsional rotations is shown to be inadequate and replaced by an alternative method which is focussed on the constituent parts of the generalized mass arising from bending and torsional displacements. Basically, the generalized mass in a particular mode is identified and examined in terms of bending, torsion and bending–torsion coupling effects. It is demonstrated that the contribution of individual components in the expression of the generalized mass of a normal mode is a much better indicator in characterizing a coupled mode. It is also shown that the usually adopted criteria of plotting bending displacement and torsional rotations to describe a coupled mode can be deceptive and misleading. In the second part of the investigation, attention is focussed on the dynamic response characteristics of bending–torsion coupled beams when subjected to random bending or torsional loads. A normal mode approach is used to establish the total response. The input random excitation is assumed to be stationary and ergodic so that with the linearity assumption, the output spectrum of the response is obtained by using the frequency response function. The contribution of each normal mode to the overall response is isolated. Particular emphasis is placed on bending-induced torsional response and torsion-induced bending response. A number of case studies involving different types of bending–torsion coupled beams with Cantilever end conditions are presented. The limitations of existing methods of modal interpretation are highlighted, and an insight into the mode selection for response analysis is provided.

1. INTRODUCTION

A considerable amount of research effort has gone into the investigation of free or forced vibration characteristics of bending–torsion coupled beams in recent years. This is because for many practical beams with cross-sections such as angle, tee, channel, etc., the vibratory motion is always coupled between bending and torsional deformations due to non-coincident centroid and shear centre. From an aeronautical point of view, perhaps, the most important example of a bending–torsion coupled beam is an aircraft wing for which

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the mass and elastic axes (which are, respectively, the loci of centroids and shear centres of the wing cross-sections) rarely coincide. Investigators of the free and forced vibration analysis of such beams recognised the important effect of the bending–torsion coupling on natural frequencies, mode shapes and response. A comprehensive review of major works on the free and forced vibration of coupled beams can be found in reference [1].

For free vibration of uncoupled beams, the interpretation of mode shapes is quite straightforward because the bending displacements and torsional rotations being uncoupled are independent, and thus can be plotted separately along the length of the beam. However, when coupling exists, the simplicity of the uncoupled case disappears because the bending displacements and torsional rotations are dependent on each other. Nevertheless, one is generally left with the same choice of plotting the bending displacements and torsional rotations together on the same graph. This traditional procedure relies on comparing the relative deformations in bending and torsional motions, which are in fact two different types of displacements with different units. Although such a procedure has been used frequently to characterize a bending–torsion coupled mode, it will be shown that on many occasions it can lead to erroneous conclusions.

From the point of view of dynamic response, the choice of a normal mode is of great importance. One of the essential purposes of this paper is to show that characterization of a coupled mode by using the conventional procedure may not be adequate in deciding the number and type of normal modes to be included in the response analysis.

By using the concept of generalized mass, a comparatively new approach in interpreting a bending–torsion coupled mode is introduced in this paper. Circumstances where commonly adopted criteria of choosing modes which could lead to wrong conclusions are identified. For free vibration characteristics, the paper deals with individual terms that contribute to the final expression of the generalized mass in a particular normal mode. Contributions of terms associated with bending, torsional and bending–torsion coupling effects to the overall generalized mass are examined for their relative importance. Then the response of the beam to a concentrated random force or torque (which is that of a white noise) is taken up by using the normal mode method and generalized co-ordinates. The constituent components of the generalized mass are used as indicative parameters in the choice of normal modes and the interpretation of response.

The theory used in this work is exact in the sense that the governing differential equations of motion are solved analytically without making any approximation enroute. For numerical results related to both free vibration and response analyses, a wide range of illustrative examples is carefully chosen for which cantilever end conditions are applied. Conclusions are drawn from the results of each example, and the shortcomings of existing procedures are outlined. Although the results are given for cantilever end condition of the beams, the theory developed is fairly general and can be easily applied to other end conditions.

2. THEORY

A bending–torsion coupled beam undergoing free natural vibration can have both its transverse and lateral displacements coupled with torsion. This generally occurs when the beam cross-section is asymmetric. The free vibration analysis of such beams has been addressed in the literature [2–4]. In the relatively simpler case when the cross-section of the beam has a single axis of symmetry, the free vibratory modes consist of bending displacements in one plane which are coupled with torsional rotation, and the bending displacements in the other plane which are de-coupled from torsion [5, 6]. A high aspect ratio aircraft wing can be idealized as a bending–torsion coupled beam [7, 8] for which the torsional coupling occurs only in one of the principal planes of bending. This is because the

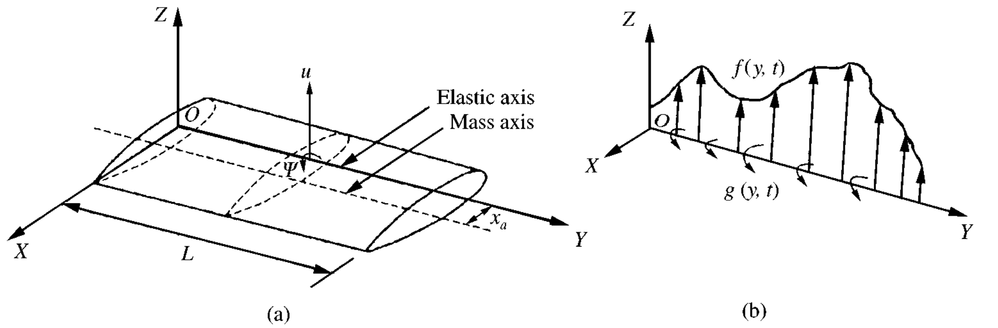

Figure 1. (a) The co-ordinate system and notation of a bending–torsion coupled beam; (b) distribution of flexural and torsional loads.

chordwise bending is usually negligible as a result of the relatively small thickness to chord ratio of the wing. From an aeroelastic point of view, such a model gives completely satisfactory results [9, 10]. In the subsequent text of this paper, attention is focussed on this simplified model of a bending–torsion coupled beam.

As an example of a bending–torsion coupled beam, a uniform aircraft wing of length L is shown in Figure 1, in a right-handed co-ordinate system. The mass and elastic axes of the wing are separated by a distance x_α as shown. It is assumed that the offset in the Z-direction (z_α) of the shear centre from the centroid of the wing cross-section is very small so that the torsional coupling in the chordwise direction is negligible. The fundamental prerequisite in response analysis put forward in this paper is based on the determination of the free vibrational modes. These are obtained by setting the external forces and damping to zero in the governing differential equations given below. The essential purpose of this paper is to present results with particular reference to generalized mass. Therefore, only a brief account of the theory is given here; a detailed description can be found in the literature [1, 11].

2.1. GOVERNING DIFFERENTIAL EQUATIONS OF MOTION

The differential equations of motion of a viscously damped uniform bending–torsion coupled beam (see Figure 1) are taken in the following form, in the same notation as in reference [1], in which a fuller details of the analysis can be found:

$$EIu'''' - c_1(\dot{u} - x_\alpha\dot{\psi}) - m(\ddot{u} - x_\alpha\ddot{\psi}) = f(y, t), \tag{1}$$

$$GJ\psi'' - c_2\dot{\psi} + c_1 x_\alpha\dot{u} - I_\alpha\ddot{\psi} + m x_\alpha\ddot{u} = g(y, t). \tag{2}$$

Here $u = u(y, t)$ and $\psi = \psi(y, t)$ are the transverse displacement and the torsional rotation of the elastic axis of the beam, respectively, $f(y, t)$ and $g(y, t)$ are the external forces and torque acting on and about the elastic axis of the beam; m is the mass per unit length, EI and GJ are, respectively, the bending and torsional rigidities of the beam, I_α is the mass moment of inertia per unit length about the elastic axis and an overdot and a prime represent differentiations with respect to time t and space y respectively.

Note that in the derivation of equations (1) and (2), elementary bending and torsion (coupled) beam theory [1, 5, 7, 8] has been used and the derivation assumes zero warping stiffness so that the torsional rigidity, torque and twist are related by the well-known St. Venant torsion theory.

The coefficients c_1 and c_2 in equations (1) and (2) are linear viscous damping terms per unit length for flexure and torsion respectively. It is assumed that each point of the cross-section moves with a different local velocity, so that in equation (1), the local damping force sums over the section to the given expression containing the c_1 term. Similarly, in equation (2) the expression containing the c_2 term is a torque about the elastic axis because of the elemental damping forces. No other sources of damping are taken into account.

2.2. FREE VIBRATION ANALYSIS

For undamped free vibration, the external load $f(y, t)$ and torque $g(y, t)$ are set to zero, together with the damping coefficients. For harmonic oscillation, the solutions for the mode shapes are then of the form [1, 5, 8]

$$U(\xi) = A_1 \cosh \alpha\xi + A_2 \sinh \alpha\xi + A_3 \cos \beta\xi + A_4 \sin \beta\xi + A_5 \cos \gamma\xi + A_6 \sin \gamma\xi, \quad (3)$$

$$\Psi(\xi) = B_1 \cosh \alpha\xi + B_2 \sinh \alpha\xi + B_3 \cos \beta\xi + B_4 \sin \beta\xi + B_5 \cos \gamma\xi + B_6 \sin \gamma\xi, \quad (4)$$

where U and Ψ are the flexural deformation and torsional rotation, A_1–A_6 and B_1–B_6 are the two different sets of constants, respectively, and

$$\xi = y/L. \quad (5)$$

α, β, γ appearing in equations (3) and (4) are functions of mechanical properties of the beam [1, 8] and are given by

$$\alpha = [2(q/3)^{1/2} \cos(\phi/3) - a/3]^{1/2}, \quad \beta = [2(q/3)^{1/2} \cos\{(\pi - \phi)/3)\} + a/3]^{1/2},$$
$$\gamma = [2(q/3)^{1/2} \cos\{(\pi + \phi)/3)\} + a/3]^{1/2}, \quad (6)$$

where

$$q = b + a^2/3, \quad \phi = \cos^{-1}[(27abc - 9ab - 2a^3)/\{2(a^2 + 3b)^{3/2}\}] \quad (7, 8)$$

with

$$a = I_\alpha \omega^2 L^2/GJ, \quad b = m\omega^2 L^4/EI, \quad c = 1 - mx_\alpha^2/I_\alpha. \quad (9)$$

The orthogonality condition of the principal modes of free vibration of the beam can be derived as [1, 11]

$$\int_0^1 [mU_m U_n + I_\alpha \Psi_m \Psi_n - mx_\alpha(U_m \Psi_n + U_n \Psi_m)] \, \mathrm{d}\xi = \mu_n \delta_{mn}, \quad (10)$$

where μ_n is the generalized mass in the nth mode and δ_{mn} is the Kronecker delta function. Note that the orthogonality condition is valid for any classical end conditions of the beam, at $\xi = 0$ and 1. From equation (10), it can be shown that the generalized mass associated with a particular mode is as follows:

$$\int_0^1 [mU_n^2 + I_\alpha \Psi_n^2 - 2mx_\alpha U_n \Psi_n] \, \mathrm{d}\xi = \mu_n. \quad (11)$$

As seen from equation (11), the expression for the generalized mass μ_n has three components. These are: mU_n^2 which is due to flexure alone, $I_\alpha \Psi_n^2$ which is due to torsion alone and $2mx_\alpha(U_n \Psi_n)$ which is due to the bending–torsion coupling effect. In each of the normal modes of vibration, the contribution of each term in the above equation is a quantitative measure of the importance of that term to generalized mass and hence to the response characteristics of that mode.

2.3. RESPONSE TO RANDOM LOADS

Randomly varying flexural and torsional excitations may be represented by [1]

$$f(\xi, t) = f(\xi)\chi_f(t) \quad \text{and} \quad g(\xi, t) = g(\xi)\chi_g(t), \tag{12}$$

where $\chi_f(t)$ and $\chi_g(t)$ are stochastic processes and their spectral densities are $S_f^\chi(\Omega)$ and $S_g^\chi(\Omega)$ respectively. The cross-spectral densities for the above loadings are

$$S_f(\xi_1, \xi_2, \Omega) = f(\xi_1)f(\xi_2)S_f^\chi(\Omega) \quad \text{and} \quad S_g(\xi_1, \xi_2, \Omega) = g(\xi_1)g(\xi_2)S_g^\chi(\Omega). \tag{13}$$

For distributed loading, the spectral density functions of the bending displacement and torsional rotation (i.e., $S_u(\xi, \Omega)$ and $S_\psi(\xi, \Omega)$) are related to the cross-spectral densities of the force $S_f(\xi_1, \xi_2, \Omega)$ and torque $S_g(\xi_1, \xi_2, \Omega)$ by the relationships [1]

$$S_u(\xi, \Omega) = \left| \sum_{n=1}^{\infty} f_n d_n(\Omega) U_n(\xi) \right|^2 S_f^\chi(\Omega) + \left| \sum_{n=1}^{\infty} g_n d_n(\Omega) U_n(\xi) \right|^2 S_g^\chi(\Omega), \tag{14}$$

$$S_\psi(\xi, \Omega) = \left| \sum_{n=1}^{\infty} f_n d_n(\Omega) \Psi_n(\xi) \right|^2 S_f^\chi(\Omega) + \left| \sum_{n=1}^{\infty} g_n d_n(\Omega) \Psi_n(\xi) \right|^2 S_g^\chi(\Omega), \tag{15}$$

in which

$$d_n(\Omega) = 1/\mu_n(\omega_n^2 - \Omega^2 + 2\mathrm{i}\zeta_n \Omega \omega_n), \tag{16}$$

where ζ_n is the non-dimensional damping coefficient in the nth mode given by [1, 11]

$$\int_0^1 [c_1 U_m U_n + c_2 \Psi_m \Psi_n - c_1 x_\alpha (U_m \Psi_n + U_n \Psi_m)] \, \mathrm{d}\xi = 2\zeta_n \omega_n \mu_n \delta_{mn} \tag{17}$$

and

$$f_n = \int_0^1 V_n(\xi) f(\xi) \, \mathrm{d}\xi, \quad g_n = \int_0^1 V_n(\xi) g(\xi) \, \mathrm{d}\xi, \tag{18}$$

where

$$V_n(\xi) = -a_F U_n(\xi_1) - a_G \Psi_n(\xi_1). \tag{19}$$

The values of a_F and a_G are either 1 or 0 depending upon whether bending or torsional loads are present or not. Finally, the mean square value of the response can be found by integrating the spectral density functions over all frequencies, so that [1]

$$\langle u^2(\xi, t) \rangle = \int_{-\infty}^{\infty} S_u(\xi, \Omega) \, \mathrm{d}\Omega, \quad \langle \psi^2(\xi, t) \rangle = \int_{-\infty}^{\infty} S_\psi(\xi, \Omega) \, \mathrm{d}\Omega. \tag{20, 21}$$

If the input random excitation follows a Gaussian probability distribution, the response probability will also be Gaussian and therefore, the response can be fully described by its spectral density function.

In the special case of white noise the power spectral density (PSD) is constant for all frequencies. Thus, S_u and S_ψ can be expressed as follows:

$$S_u(\xi, \Omega) = S_\psi(\xi, \Omega) = S_0, \tag{22}$$

where S_0 is a constant.

3. DISCUSSION OF RESULTS

Results are given for four wide ranging bending–torsion coupled beams with Cantilever end conditions. These examples which are available in the published literature, include two aircraft wings, a box beam with an axial slit and a thin-walled beam with semi-circular cross-section. All of them exhibit bending–torsion coupling due to non-coincident mass and shear centres. For simplicity these beams have been numbered sequentially from ① to ④: ① is for the Goland wing [9], ② is for the Loring wing [10], ③ is for the box beam with an axial slit [8] and ④ is for the thin-walled beam with semi-circular cross-section [3]. The basic data used in the analysis for these beams are given in Table 1.

The following results are given for each of the coupled beams: (a) the first six natural frequencies and mode shapes; (b) comparison of generalized mass in each coupled mode with the generalized mass in pure (uncoupled) bending or torsional displacements; (c) contribution of each component (bending, torsion and coupling) to the generalized mass in each of the normal modes; (d) contribution of each mode to the overall response of the beam at the tip due to a unit flexural load at the tip; (e) modal contribution to the overall response of the beam at the tip due to a unit torque at the tip. The above five sets of results are categorized into a series (a)–(e), proceeded by the corresponding figure number for the particular beam under investigation.

The generalized mass associated with uncoupled free vibration of such beams [12] in pure bending is mL/4 (kg m^2) and in pure torsion is $I_\alpha L/2$ (kg m^2) and the numerical values for the particular beams investigated are given in Table 2. The wide ranging values of the generalized mass in uncoupled bending and torsional modes for these illustrative examples have been chosen to demonstrate the versatile application of the proposed method. The dimensions shown are kg m^2 since the bending deflection of the tip is taken to be in meter. For a bending–torsion coupled beam, the associated generalized mass in a normal mode (coupled in bending and torsion) will not be equal to either of these two values, due to the bending–torsion coupling effect. Moreover, in undamped and uncoupled free vibration of beams, generalized masses for each of the flexural and torsional modes are constant in all modes, but — as will be shown later — these values will be significantly altered in each mode for bending–torsion coupled beams due to the coupling effect.

TABLE 1

Structural properties of example beams coupled in bending and torsion

Type of example	EI (N m^2)	GJ (N m^2)	m (kg/m)	I_α (kg m)	x_α (m)	L (m)
Goland wing [9]	$9{\cdot}7567 \times 10^6$	$9{\cdot}88 \times 10^5$	35·75	8·65	0·183	6·096
Loring wing [10]	677·6	1019	8·06	0·0585	0·038	2·06
Box beam with an axial slit [8]	$5{\cdot}8 \times 10^4$	78·3	2·45	0·02	0·08	5·0
Thin-walled semi-circle [3]	6380·14	43·46	0·835	0·000501	0·0155	0·82

TABLE 2

Generalized mass for the uncoupled case of the example beams shown in Table 1

	Generalized mass	
	Pure bending	Pure torsion
Example beams	$mL/4$ (kg m^2)	$I_\alpha L/2$ (kg m^2)
Goland wing [9]	54·483	23·365
Loring wing [10]	4·1509	0·060255
Box beam with an axial slit [8]	3·0625	0·05
Semi-circle, thin-walled [3]	0·17118	$2{\cdot}0541 \times 10^{-4}$

Results of the investigation for all the coupled beams numbered ①–④ are shown in Figures 2–5, respectively, and the following observations are made when interpreting the results.

3.1. FREE VIBRATION ANALYSIS

Figures 2(a)–(5a) show the first six natural frequencies and mode shapes for bending displacements (U) and torsional rotation (Ψ) of the each of the four example beams, respectively, whilst Figures 2(b)–(5b) show the generalized mass associated with each of the six modes together with the constant values for the uncoupled case. Figures (2c)–(5c) show, in a histogram, the percentage contributions of individual terms in equation (11) to the overall generalized mass. These figures were critically examined to designate the true nature of a coupled mode. The following general principles were applied in interpreting the modes depicted by these three categories of results, namely the figures of series (a)–(c).

For series (a), bending displacements (U) are directly compared with torsional rotations (Ψ), even though their units are different. In series (b) the total generalized mass associated with each mode is shown in a bar chart together with its constant value for the uncoupled case in bending and torsional free vibration (see horizontal lines shown in the figures). The series (c) figures show contributory terms as a percentage resulting from bending, torsion and coupling effects which altogether sum to the total generalized mass. The figures in this particular series have been drawn using different patterns, so that when all the individual contributions are added up the net (100%) generalized mass is obtained.

These three different ways of examining a coupled mode are particularly relevant to compare one method with another. For series (a) figures, a straightforward comparison is made to determine whether a mode is bending, torsional or coupled. The series (b) figures reveal the nature of a normal mode in a very different way because the total generalized mass in a mode has been used as the indicative parameter in relation to its value for the uncoupled case. Here the proximity of the actual generalized mass to the uncoupled value, either bending or torsion, is taken as the criterion to interpret a mode. Finally, in series (c) figures, the intrinsic contribution of bending, torsion and coupling terms to the total generalized mass in each mode is taken into account to illustrate the nature of the mode.

For presentational purposes, each of the normal modes is identified in the following way: (i) a mode with mostly bending displacements and with negligible torsional rotation, referred to as **B**, (ii) a mode with mostly torsional rotation and with negligible bending

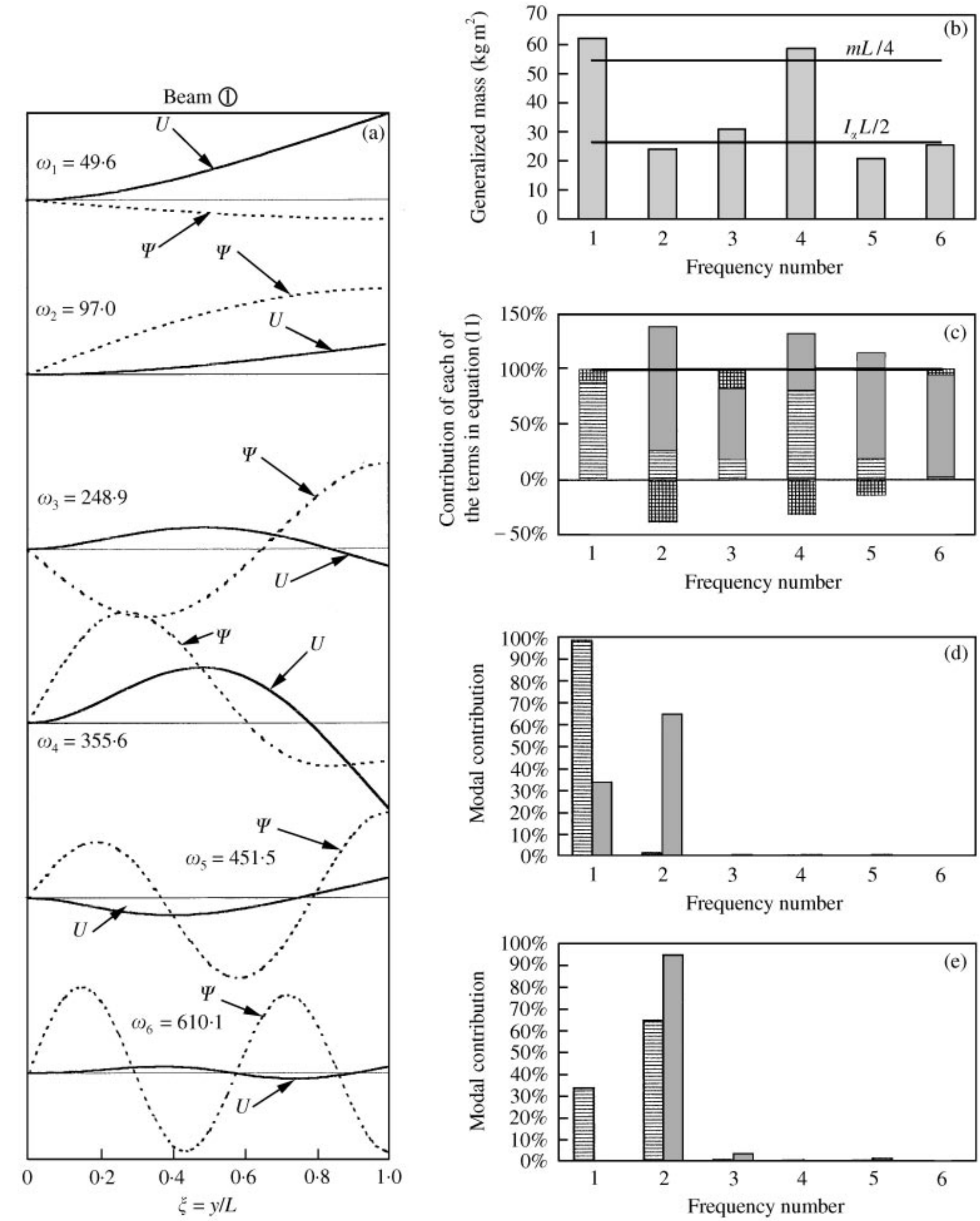

Figure 2. Significance of generalized mass in the dynamic response characteristics of a bending–torsion coupled beam using the example of Goland wing, (a) The first six natural frequencies (rad/s) and mode shapes;(b) comparison of generalized mass in each bending–torsion coupled mode with generalized mass in pure flexural or torsional modes; (c) contribution of each term (bending, torsion and coupling) to the generalized mass in different modes (▤, bending; ■, torsion; ▦, coupling); (d) percentage of modal contribution in the dynamic flexural and torsional response of the beam at the tip due to the flexural load at the tip (▤, bending; ■, torsion); (e) percentage of modal contribution in the dynamic flexural and torsional response of the beam at the tip due to the torque at the tip (▤, bending; ■, torsion).

displacements, referred to as **T**, (iii) a mode with predominant bending displacements and with some appreciable torsional rotations resulting from light to moderate coupling, referred to as $\mathbf{B_C}$, (iv) a mode with predominant torsional rotations and with some

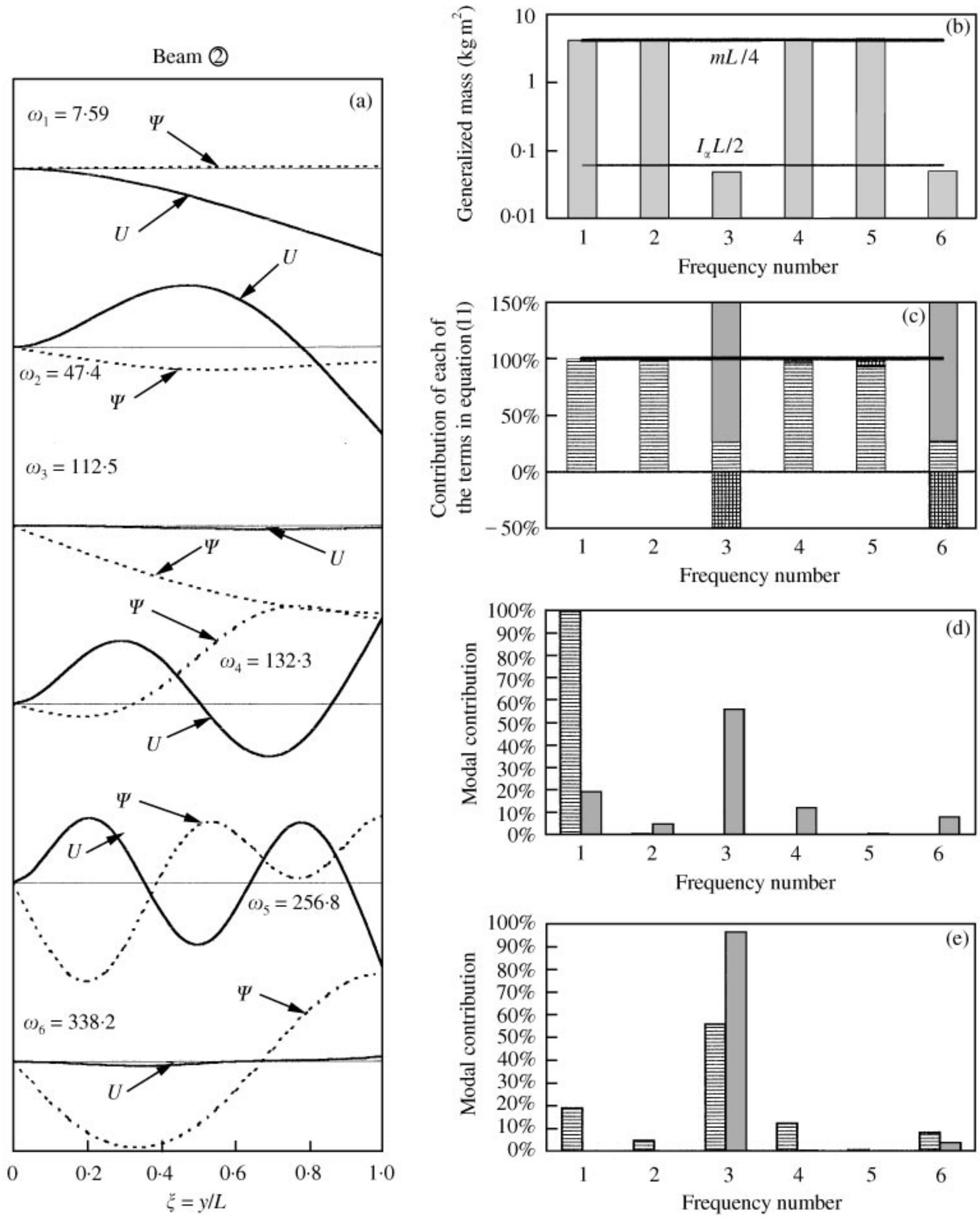

Figure 3. Significance of generalized mass in the dynamic response characteristics of a bending–torsion coupled beam using the example of Loring wing, (a) the first six natural frequencies (rad/s) and mode shapes; (b) comparison of generalized mass in each bending–torsion coupled mode with generalized mass in pure flexural or torsional modes; (c) contribution of each term (bending, torsion and coupling) to the generalized mass in different modes (▤, bending; ▩, torsion; ▦, coupling); (d) percentage of modal contribution in the dynamic flexural and torsional response of the beam at the tip due to the flexural load at the tip (▤, bending; ▩, torsion); (e) percentage of modal contribution in the dynamic flexural and torsional response of the beam at the tip due to the torque at the tip, (▤,bending; ▩, torsion).

appreciable bending displacements resulting from light to moderate coupling, referred to as $\mathbf{T_C}$, and (v) a mode in which bending displacements and torsional rotations play more or less similar roles, referred to as a heavily coupled mode **C**.

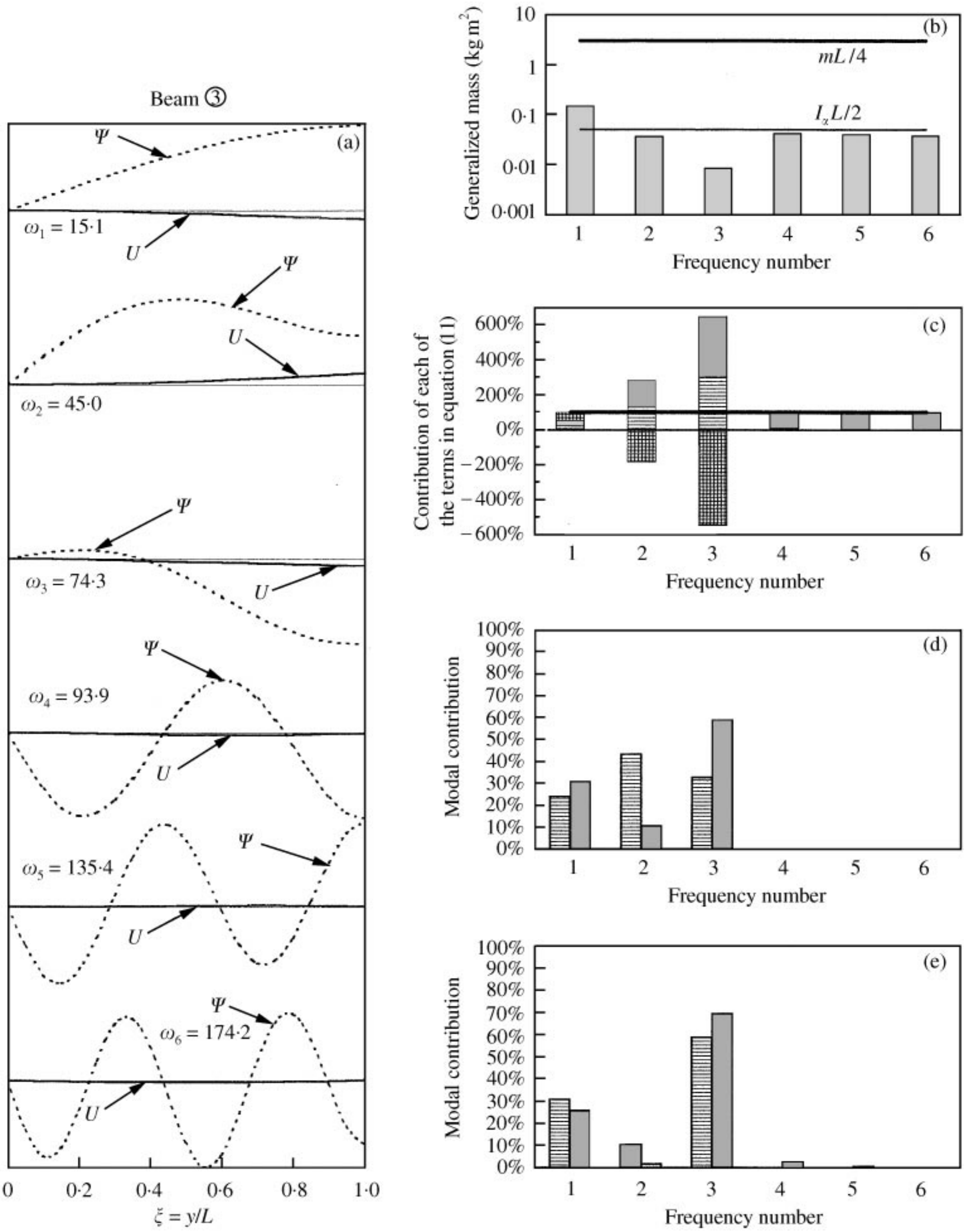

Figure 4. Significance of generalized mass in the dynamic response characteristics of a bending–torsion coupled beam using the example of a box beam with an axial slit, (a) the first six natural frequencies (rad/s) and mode shapes, (b) comparison of generalized mass in each bending–torsion coupled mode with generalized mass in pure flexural or torsional modes; (c) contribution of each term (bending, torsion and coupling) to the generalized mass in different modes (▤, bending; ■, torsion; ▦, coupling); (d) percentage of modal contribution in the dynamic flexural and torsional response of the beam at the tip due to the flexural load at the tip (▤, bending; ■, torsion); (e) percentage of modal contribution in the dynamic flexural and torsional response of the beam at the tip due to the torque at the tip, (▤, bending; ■, torsion).

With the above classification, the results for series (a) to series (c) have been summarized in Table 3. The three main types of characterization used often lead to the same conclusion as to whether a mode is bending, torsional or coupled as clear from Table 3. However, there

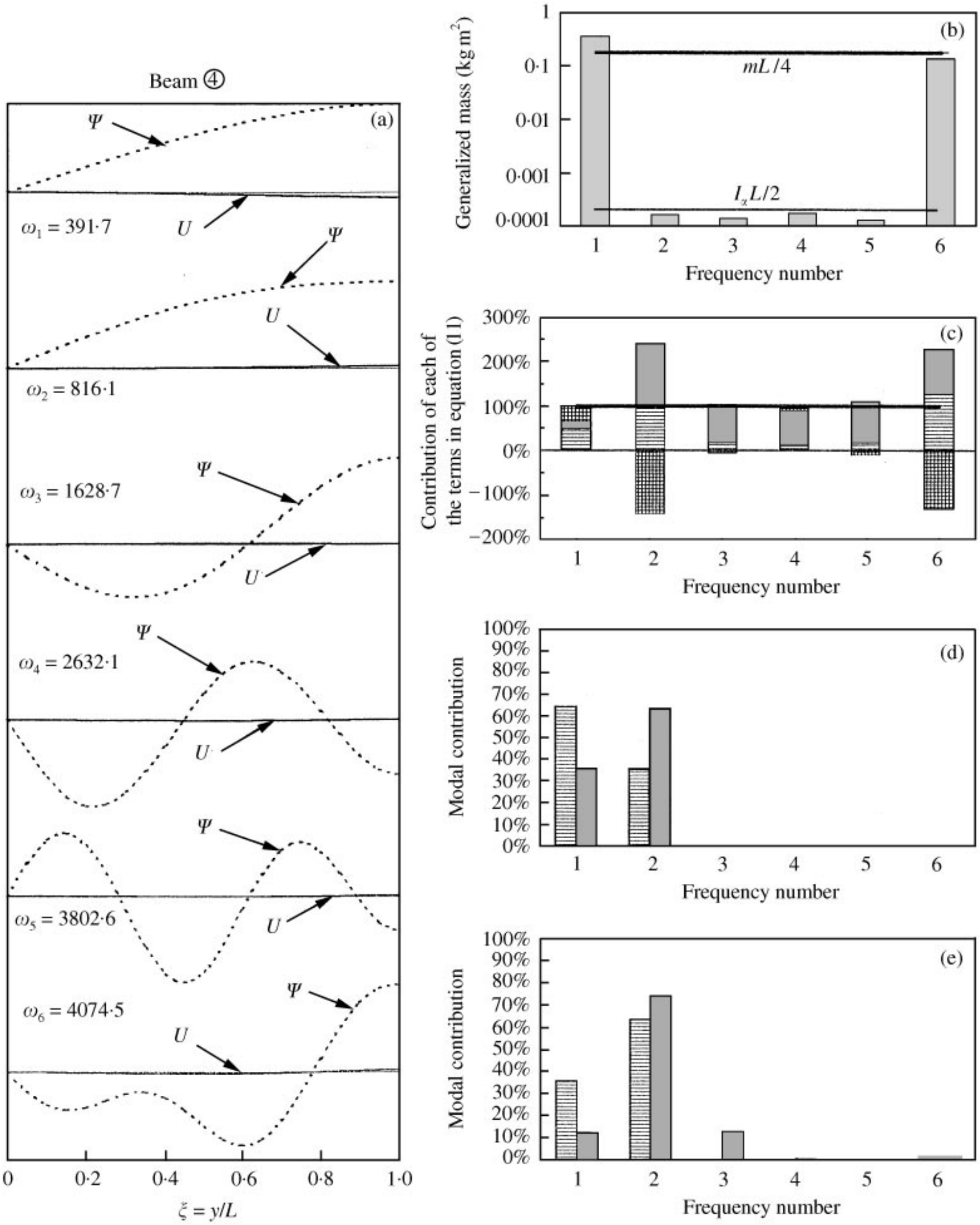

Figure 5. Significance of generalized mass in the dynamic response characteristics of a bending–torsion coupled beam using the example of a thin-walled semi-circular cross-section, (a) the first six natural frequencies (rad/s) and mode shapes; (b) comparison of generalized mass in each bending–torsion coupled mode with generalized mass in pure flexural or torsional modes; (c) contribution of each term (bending, torsion and coupling) to the generalized mass in different modes (▤, bending; ■, torsion; ▦, coupling); (d) percentage of modal contribution in the dynamic flexural and torsional response of the beam at the tip due to the flexural load at the tip (▤, bending; ■, torsion); (e) percentage of modal contribution to the dynamic flexural and torsional response of the beam at the tip due to the torque at the tip, (▤, bending; ■, torsion).

are cases where the three criteria used lead to three different conclusions. These are shown in bold script in Table 3. For example, the fifth mode of the beam ② shown in Figure 3(a) indicates that it is a heavily coupled mode (**C**). In contrast, Figures 3(b) show that the fifth

TABLE 3

Characterization criteria of modes based on displacement, total generalized mass and contributory generalized mass

		Characterization criteria		
Beam no.	Mode	Displacement	Total generalized mass	Contributory generalized mass
①	1	B_C	B_C	B_C
	2	T_C	T_C	T_C
	3	T_C	T_C	T_C
	4	**C**	$\mathbf{B}_C$	**C**
	5	T_C	T_C	T_C
	6	T	T	T
②	1	B	B	B
	2	B_C	B	B
	3	T	T_C	T_C
	4	**C**	**B**	**B**
	5	**C**	**B**	$\mathbf{B}_C$
	6	**T**	$\mathbf{T}_C$	$\mathbf{T}_C$
③	**1**	$\mathbf{T}_C$	$\mathbf{T}_C$	**C**
	2	$\mathbf{T}_C$	$\mathbf{T}_C$	**C**
	3	$\mathbf{T}_C$	$\mathbf{T}_C$	**C**
	4	T	T	T
	5	T	T	T
	6	T	T	T
④	**1**	$\mathbf{T}_C$	$\mathbf{B}_C$	$\mathbf{B}_C$
	2	**T**	$\mathbf{T}_C$	**C**
	3	**T**	$\mathbf{T}_C$	$\mathbf{T}_C$
	4	**T**	$\mathbf{T}_C$	$\mathbf{T}_C$
	5	**T**	$\mathbf{T}_C$	$\mathbf{T}_C$
	6	**T**	$\mathbf{B}_C$	**C**

mode is a bending mode (**B**), but, when judged from the results of Figure 3(c) it is predominantly a bending mode with a small amount of torsion in it ($\mathbf{B}_C$). There are other examples in the table for which the simple-minded rule based on displacements may lead to misleading conclusions. For instance, the fourth mode of beam ①, in Figure 2(a) based on displacements and rotations shows that the mode is heavily coupled in bending and torsion (**C**) whereas Figure 2(b) which is based on the total generalized mass in that mode, reveals a different picture. Figure 2(b) indicates that the total generalized mass, being very close to that of the uncoupled case of free bending vibration can be classified as a predominantly bending mode with a small amount of torsion in it ($\mathbf{B}_C$). Interestingly, Figure 2(c) which is based on contributory generalized mass, leads to the same conclusion as Figure 2(a) showing that the mode is heavily coupled (**C**). The investigation has shown that if a choice is to be made between the criteria of total generalized mass and contributory generalized mass, it is the latter which comes more and more into play to reveal the true nature of a bending–torsion coupled mode.

3.2. RESPONSE ANALYSIS

Following the modal analysis, a response study is undertaken. Each cantilever beam is subjected in turn to a unit transverse force or to a unit torque at the tip, which is assumed to be random with the PSD distribution of an ideal white noise. The convergence of results for response analysis was assured by using enough number of modes and it was found that the first six normal modes were sufficient for each of the four illustrative beams. The response quantities computed, are the mean-square values of the bending displacement and torsional rotation of the cantilever beam at the tip.

For all the above cantilever beams, Figures 2(d)–5(d) show the percentage of modal contribution to both bending and torsional response arising from a unit external transverse excitation of white noise at the tip. Similarly, Figures. 2(e)–5(e) show the percentage of modal contribution to both bending and torsional response arising from a unit external torsional excitation of white noise at the tip. Both sets of results show that the coupling effect in all the four example beams is pronounced showing the occurrence of bending-induced twist and torsion-induced bending displacements. There are some interesting features of these results, which are self-explanatory, but the ones which draw special attention, need emphasising.

Figures 2(d)–5(d) show that the first few bending-dominated modes have made the largest contribution to the dynamic flexural displacement for all the beams, whereas Figures 2(e)–5(e) show that the first few torsion-dominated modes have made the largest contribution to the dynamic torsional rotation for all the beams.

In the case of beam ① only the first mode contributes to the flexural response due to the transverse force, but the torsional response due to this transverse force is induced by the first two modes with a greater contribution from the second mode. Only the second mode contributes to the torsional response due to the external torque, but the bending response due to this torque is induced by the first two modes with a greater contribution from the second mode. This is expected and is in accordance with the observation made in Table 3, using all three criteria.

A similar trend for the bending response is observed for beam ② when subjected to a transverse force. However, the third mode contributes most to bending-induced torsional response with relatively smaller contributions from the first, fourth, sixth and second modes respectively. In relation to the torsional response due to the external torque, it is clearly evident that the most important mode which contributes mainly to the response, is the third mode. However, the torsion-induced bending response is mostly due to the third, first, and fourth modes predominantly. Here the criterion of contributory generalized mass as illustrated in Table 3 provides a logical explanation of these results.

For beam ③ the bending response as well as the bending-induced torsional response due to the transverse force, are principally governed by the first, second and third modes, respectively. On the other hand, the torsional response as well as the torsion-induced bending response due to the external torque, are mainly influenced by the first and third modes, with a small contribution from the second mode. The importance of characterizing the modes in terms of contributory generalized mass is particularly evident from the results shown in Table 3.

Finally, for beam ④, the bending response as well as the bending-induced torsional response due to the transverse force, are principally governed by the first and second modes respectively. Similarly, the torsional response as well as the torsion-induced bending response due to the external torque, are mainly influenced by the first and second modes, with a small amount of contribution from the third mode. Here again, the characterisation of modes in terms of contributory generalized mass is seen to be the best way of assessing the modes (see Table 3).

4. CONCLUSIONS

Three different but related methods of characterizing normal modes of bending–torsion coupled beams have been studied in detail with particular reference to the concept of generalized mass. The constituent terms which contribute to the overall generalized mass of a bending–torsion coupled beam have been identified for a better understanding of a coupled mode. Numerical results obtained for a wide range of coupled beams with different applications show that traditional methods of modal identification solely based on displacement, can be misleading. The investigation has reaffirmed that due recognition to the significance of generalized mass must be given when studying the free and forced vibration characteristics of bending–torsion coupled beams. Although the method has been applied to the special case of cantilever beams when obtaining numerical results, it is quite straightforward to use for other end conditions.

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